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Holographic Reconstruction of Spacetime and Renormalization in the AdS/CFT Correspondence

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Abstract

We develop a systematic method for renormalizing the AdS/CFT prescription for computing correlation functions. This involves regularizing the bulk on-shell supergravity action in a covariant way, computing all divergences, adding counterterms to cancel them and then removing the regulator. We explicitly work out the case of pure gravity up to six dimensions and of gravity coupled to scalars. The method can also be viewed as providing a holographic reconstruction of the bulk spacetime metric and of bulk fields on this spacetime, out of conformal field theory data. Knowing which sources are turned on is sufficient in order to obtain an asymptotic expansion of the bulk metric and of bulk fields near the boundary to high enough order so that all infrared divergences of the on-shell action are obtained. To continue the holographic reconstruction of the bulk fields one needs new CFT data: the expectation value of the dual operator. In particular, in order to obtain the bulk metric one needs to know the expectation value of stress-energy tensor of the boundary theory. We provide completely explicit formulae for the holographic stress-energy tensors up to six dimensions. We show that both the gravitational and matter conformal anomalies of the boundary theory are correctly reproduced. We also obtain the conformal transformation properties of the boundary stress-energy tensors.

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1 Introduction and summary of the results

Holography states that a $(d+1)$-dimensional gravitational theory (referred to as the bulk theory) should have a description in terms of a d-dimensional field theory (referred to as the boundary theory) with one degree of freedom per Planck area [45, 41]. The arguments leading to the holographic principle use rather generic properties of gravitational physics, indicating that holography should be a feature of any quantum theory of gravity. Nevertheless it has been proved a difficult task to find examples where holography is realized, let alone to develop a precise dictionary between bulk and boundary physics. The AdS/CFT correspondence [31] provides such a realization [47, 42] with a rather precise computational framework [24, 47]. It is, therefore, desirable to sharpen the existing dictionary between bulk/boundary physics as much as possible. In particular, one of the issues one would like to understand is how spacetime is built holographically out of field theory data.

The prescription of [24, 47] gives a concrete proposal for a holographic computation of physical observables. In particular, the partition function of string theory compactified on AdS spaces with prescribed boundary conditions for the bulk fields is equal to the generating functional of conformal field theory correlation functions, the boundary value of fields being now interpreted as sources for operators of the dual conformal field theory (CFT). String theory on anti-de Sitter (AdS) spaces is still incompletely understood. At low energies, however, the theory becomes a gauged supergravity with an AdS ground state coupled to Kaluza-Klein (KK) modes. On the field theory side, this corresponds to the large N and strong 't Hooft coupling regime of the CFT. So in the AdS/CFT context the question is how one can reconstruct the bulk spacetime out of CFT data. One can also pose the converse question: given a bulk spacetime, what properties of the dual CFT can one read off?

The prescription of [24, 47] equates the on-shell value of the supergravity action with the generating functional of connected graphs of composite operators. Both sides of this correspondence, however, suffer from infinities —infrared divergences on the supergravity side and ultraviolet divergences on the CFT side. Thus, the prescription of [24, 47] should more properly be viewed as an equality between bare quantities. Ones needs to renormalize the theory to obtain a correspondence between finite quantities. It is one of the aims of this paper to present a systematic way of performing such renormalization.

The CFT data[4] that we will use are: which operators are turned on, and what is their vacuum expectation value. Since the boundary metric (or, more properly, the boundary conformal structure) couples to the boundary stress-energy tensor, the reconstruction of the bulk metric to leading order involves a detailed knowledge of the way the energy-momentum tensor is encoded holographically. There is by now an extended literature on the study of the stress-energy tensor in the context of the AdS/CFT correspondence starting from [3, 34]. We will build on these and other related works [15, 32, 30]. Our starting point will be the calculation of the infrared divergences of the on-shell gravitational action [26]. Minimally subtracting the divergences by adding counterterms [26] leads straightforwardly to the results in [3, 15, 30]. After the subtractions have been made one can remove the (infrared) regulator and obtain a completely explicit formula for the expectation value of the dual stress-energy tensor in terms of the gravitational solution.

We will mostly concentrate on the gravitational sector, i.e. in the reconstruction of the bulk metric, but we will also discuss the coupling to scalars. Our approach will be to build perturbatively an Einstein

[4] We assume that the CFT we are discussing has an AdS dual. Our results only depend on the spacetime dimension and apply to all cases where the AdS/CFT duality is applicable, so we shall not specify any particular CFT model.

manifold of constant negative curvature (which we will sometimes refer to as an asymptotically AdS space) as well as a solution to the scalar field equations on this manifold out of CFT data. The CFT data we start from is what sources are turned on. We will include a source for the dual stress-energy tensor as well as sources for scalar composite operators. This means that in the bulk we need to solve the gravitational equations coupled to scalars given a conformal structure at infinity and appropriate Dirichlet boundary conditions for the scalars. It is well-known that if one considers the standard Euclidean AdS (i.e., with isometry $SO(1, d+1)$), the scalar field equation with Dirichlet boundary conditions has a unique solution. In the Lorentzian case, because of the existence of normalizable modes, the solution ceases to be unique. Likewise, the Dirichlet boundary condition problem for (Euclidean) gravity has a unique (up to diffeomorphisms) smooth solution in the case the bulk manifold in topologically a ball and the boundary conformal structure sufficiently close to the standard one [21]. However, given a boundary topology there may be more than one Einstein manifold with this boundary. For example, if the boundary has the topology of $S^1 \times S^{d-1}$, there are two possible bulk manifolds [25, 47]: one which is obtained from standard AdS by global identifications and is topologically $S^1 \times R^d$, and another, the Schwarzschild-AdS black hole, which is topologically $R^2 \times S^{d-1}$.

We will make no assumption on the global structure of the space or on its signature. The CFT should provide additional data in order to retrieve this information. Indeed, we will see that only the information about the sources leaves undetermined the part of the solution which is sensitive on global issues and/or the signature of spacetime. To determine that part one needs new CFT data. To leading order these are the expectation values of the CFT operators.

In particular, in the case of pure gravity, we find that generically a boundary conformal structure is not sufficient in order to obtain the bulk metric. One needs more CFT data. To leading order one needs to specify the expectation value of the boundary stress-energy tensor. Since the gravitational field equation is a second order differential equation, one may expect that these data are sufficient in order to specify the full solution. In general, however, non-local observables such as Wilson loops may be needed in order to recover global properties of the solution and reconstruct the metric in the deep interior region. Furthermore, higher point functions of the stress-energy tensor may be necessary if higher derivatives corrections such as R^2 terms are included in the action. We emphasize that we make no assumption about the regularity of the solution. Under additional assumptions the metric may be determined by fewer data. For example, as we mentioned above, under certain assumptions on the topology and the boundary conformal structure one obtains a unique smooth solution [21]. Another example is the case when one restricts oneself to conformally flat bulk metrics. Then a conformally flat boundary metric does yield a unique, up to diffeomorphisms and global identifications, bulk metric [40].

Turning things around, given a specific solution, we present formulae for the expectation values of the dual CFT operators. In particular, in the case the operator is the stress energy tensor, our formulae have a "dual" meaning [3]: both as the expectation value of the stress-energy tensor of the dual CFT and as the quasi-local stress-energy tensor of Brown and York [11]. We provide very explicit formulae for the stress-energy tensor associated with any solution of Einstein's equations with negative constant curvature.

Let us summarize these results for spacetime dimension up to six. The first step is to rewrite the solution in the Graham-Fefferman coordinate system [16]

$$ds^2 = G_{\mu\nu} dx^\mu dx^\nu = \frac{l^2}{r^2} \left(dr^2 + g_{ij}(x, r) dx^i dx^j \right) \tag{1.1}$$

where

$$g(x,r) = g_{(0)} + r^2 g_{(2)} + \cdots + r^d g_{(d)} + h_{(d)} r^d \log r^2 + \mathcal{O}(r^{d+1}) \tag{1.2}$$

The logarithmic term appears only in even dimensions and only even powers of r appear up to order $r^{[(d-1)]}$, where $[a]$ indicates the integer part of a. l is a parameter of dimension of length related to the cosmological constant as $\Lambda = -\frac{d(d-1)}{2l^2}$. Any asymptotically AdS metric can be brought in the form (1.1) near the boundary ([21], see also [22, 20]). Once this coordinate system has been reached, the stress-energy tensor reads

$$\langle T_{ij} \rangle = \frac{dl^{d-1}}{16\pi G_{\rm N}} g_{(d)ij} + X_{ij}[g_{(n)}]. \tag{1.3}$$

where $X_{ij}[g_{(n)}]$ is a function of $g_{(n)}$ with $n < d$. Its exact form depends on the spacetime dimension and it reflects the conformal anomalies of the boundary conformal field theory. In odd (boundary) dimensions, where there are no gravitational conformal anomalies, X_{ij} is equal to zero. The expression for $X_{ij}[g_{(n)}]$ for $d = 2, 4, 6$ can be read off from (3.10), (3.15) and (3.16), respectively. The universal part of (1.3) (i.e. with X_{ij} omitted) was obtained previously in [34]. Actually, to obtain the dual stress-energy tensor it is sufficient to only know $g_{(0)}$ and $g_{(d)}$ as $g_{(n)}$ with $n < d$ are uniquely determined from $g_{(0)}$, as we will see. The coefficient $h_{(d)}$ of the logarithmic term in the case of even d is also directly related to the conformal anomaly: it is proportional to the metric variation of the conformal anomaly.

It was pointed out in [3] that this prescription for calculating the boundary stress-energy tensor provides also a novel, free of divergences[5], way of computing the gravitational quasi-local stress-energy tensor of Brown and York [11]. This approach was recently criticized in [2], and we take this opportunity to address this criticism. Conformal anomalies reflect infrared divergences in the gravitational sector [26]. Because of these divergences one cannot maintain the full group of isometries even asymptotically. In particular, the isometries of AdS that rescale the radial coordinate (these correspond to dilations in the CFT) are broken by infrared divergences. Because of this fact, bulk solutions that are related by diffeomorphisms that yield a conformal transformation in the boundary do not necessarily have the same mass. Assigning zero mass to the spacetime with boundary R^d, one obtains that, due to the conformal anomaly, the solution with boundary $R \times S^{d-1}$ has non-zero mass. This parallels exactly the discussion in field theory. In that case, starting from the CFT on R^d with vanishing expectation value of the stress-energy tensor, one obtains the Casimir energy of the CFT on $R \times S^{d-1}$ by a conformal transformation [12]. The agreement between the gravitational ground state energy and the Casimir energy of the CFT is a direct consequence of the fact that the conformal anomaly computed by weakly coupled gauge theory and by supergravity agree [26]. It should be noted that, as emphasized in [3], agreement between gravity/field theory for the ground state energy is achieved only after all ambiguities are fixed in the same manner on both sides.

A conformal transformation in the boundary theory is realized in the bulk as a special diffeomorphism that preserves the form of the coordinate system (1.1) [28]. Using these diffeomorphisms one can easily study how the (quantum, i.e., with the effects of the conformal anomaly taken into account) stress-energy tensor transforms under conformal transformations. Our results, when restricted to the cases studied in the literature [12], are in agreement with them. We note that the present determination is considerably easier than the one in [12].

The discussion is qualitatively the same when one adds matter to the system. We discuss scalar fields

[5]We emphasize, however, that one has to subtract the logarithmic divergences in even dimensions in order for the stress-energy tensor to be finite.

but the discussion generalizes straightforwardly to other kinds of matter. We study both the case the gravitational background is fixed and the case gravity is dynamical.

Let us summarize the results for the case of scalar fields in a fixed gravitational background (given by a metric of the form (1.1)). We look for solutions of massive scalar fields with mass $m^2 = (\Delta - d)\Delta$ that near the boundary have the form (in the coordinate system (1.1))

$$\Phi(x, r) = r^{d-\Delta} \left(\phi_{(0)} + r^2 \phi_{(2)} + \cdots + r^{2\Delta-d} \phi_{(2\Delta-d)} + r^{2\Delta-d} \log r^2 \psi_{(2\Delta-d)}\right) + \mathcal{O}(r^{\Delta+1}). \tag{1.4}$$

The logarithmic terms appears only when $2\Delta - d$ is an integer and we only consider this case in this paper. We find that $\phi_{(n)}$, with $n < 2\Delta - d$, and $\psi_{(2\Delta-d)}$ are uniquely determined from the scalar field equation. This information is sufficient for a complete determination of the infrared divergences of the on-shell bulk action. In particular, the logarithmic term $\psi_{(2\Delta-d)}$ in (1.4) is directly related to matter conformal anomalies. These conformal anomalies were shown not to renormalize in [37]. We indeed find exact agreement with the computation in [37]. Adding counterterms to cancel the infrared divergences we obtain the renormalized on-shell action. We stress that even in the case of a free massive scalar field in a fixed AdS background one needs counterterms in order for the on-shell action to be finite (see (5.9)). The coefficient $\phi_{(2\Delta-d)}$ is left undetermined by the field equations. It is determined, however, by the expectation value of the dual operator. Differentiating the renormalized on-shell action one finds (up to terms contributing contact terms in the 2-point function)

$$\langle O(x) \rangle = (2\Delta - d)\phi_{(2\Delta-d)}(x) \tag{1.5}$$

This relation, with the precise proportionality coefficient, has first been derived in [29]. The value of the proportionality coefficient is crucial in order to obtain the correct normalization of the 2-point function in standard AdS background [17].

In the case the gravitational background is dynamical we find that, for scalars that correspond to irrelevant operators, our perturbative treatment is consistent only if one considers single insertions of the irrelevant operator, i.e. the source is treated as an infinitesimal parameter, in agreement with the discussion in [47]. For scalars that correspond to marginal and relevant operators one can compute perturbatively the back-reaction of the scalars to the gravitational background. One can then regularize and renormalize as in the discussion of pure gravity or scalars in a fixed background. For illustrative purposes we analyze a simple example.

This paper is organized as follows. In the next section we discuss the Dirichlet problem for AdS gravity and we obtain an asymptotic solution for a given boundary metric (up to six dimensions). In section 3 we use these solutions to obtain the infrared divergences of the on-shell gravitational action. After renormalizing the on-shell action by adding counterterms, we compute the holographic stress-energy tensor. Section 4 is devoted to the study of the conformal transformation properties of the boundary stress-energy tensor. In section 5 we extend the analysis of sections 2 and 3 to include matter. In appendices A and D we give the explicit form of the solutions discussed in section 2 and section 5. Appendix B contains the explicit form of the counterterms discussed in section 3. Finally, in appendix C we present a proof that the coefficient of the logarithmic term in the metric (present in even boundary dimensions) is proportional to the metric variation of the conformal anomaly.

2 Dirichlet boundary problem for AdS gravity

The Einstein-Hilbert action for a theory on a manifold M with boundary ∂M is given by[6]

$$S_{\rm gr}[G] = \frac{1}{16\pi G_{\rm N}}[\int_M \mathrm{d}^{d+1}x\,\sqrt{G}\,(R[G]+2\Lambda) - \int_{\partial M}\mathrm{d}^d x\,\sqrt{\gamma}\,2K], \tag{2.1}$$

where K is the trace of the second fundamental form and γ is the induced metric on the boundary. The boundary term is necessary in order to get an action which only depends on first derivatives of the metric [18], and it guarantees that the variational problem with Dirichlet boundary conditions is well-defined.

According to the prescription of [24, 47], the conformal field theory effective action is given by evaluating the on-shell action functional. The field specifying the boundary conditions for the metric is regarded as a source for the boundary operator. We therefore need to obtain solutions to Einstein's equations,

$$R_{\mu\nu} - \frac{1}{2}RG_{\mu\nu} = \Lambda G_{\mu\nu}, \tag{2.2}$$

subject to appropriate Dirichlet boundary conditions.

Metrics $G_{\mu\nu}$ that satisfy (2.2) have a second order pole at infinity. Therefore, they do not induce a metric at infinity. They do induce, however, a conformal class, i.e. a metric up to a conformal transformation. This is achieved by introducing a defining function r, i.e. a positive function in the interior of M that has a single zero and non-vanishing derivative at the boundary. Then one obtains a metric at the boundary by $g_{(0)} = r^2 G|_{\partial M}$ [7]. However, any other defining function $r' = r \exp w$ is as good. Therefore, the metric $g_{(0)}$ is only defined up to a conformal transformation.

We are interested in solving (2.2) given a conformal structure at infinity. This can be achieved by working in the coordinate system (1.1) introduced by Feffermam and Graham [16]. The metric in (1.1) contains only even powers of r up to the order we are interested in [16] (see also [22, 20]). For this reason, it is convenient to use the variable $\rho = r^2$ [26], [8]

$$\begin{aligned} ds^2 &= G_{\mu\nu}dx^\mu dx^\nu = l^2\left(\frac{d\rho^2}{4\rho^2} + \frac{1}{\rho}g_{ij}(x,\rho)dx^i dx^j\right) \\ g(x,\rho) &= g_{(0)} + \cdots + \rho^{d/2}g_{(d)} + h_{(d)}\rho^{d/2}\log\rho + \ldots \end{aligned} \tag{2.3}$$

where the logarithmic piece appears only for even d. The sub-index in the metric expansion (and in all other expansions that appear in this paper) indicates the number of derivatives involved in that term, i.e. $g_{(2)}$ contains two derivatives, $g_{(4)}$ four derivatives, etc. It follows that the perturbative expansion in ρ is also a low energy expansion. We set $l = 1$ from now on. One can easily reinstate the factors of l by dimensional analysis.

[6] Our curvature conventions are as follows $R_{ijk}{}^l = \partial_i \Gamma_{jk}{}^l + \Gamma_{ip}{}^l\Gamma_{jk}{}^p - i \leftrightarrow j$ and $R_{ij} = R_{ikj}{}^k$. We these conventions the curvature of AdS comes out positive, but we will still use the terminology "space of constant negative curvature". Notice also that we take $\int \mathrm{d}^{d+1}x = \int \mathrm{d}^d x \int_0^\infty \mathrm{d}r$ and the boundary is at $r = 0$ (in the coordinate system (1.1)). The minus sign in front of the trace of the second fundamental form is correlated with the choice of having $r = 0$ in the lower end of the radial integration.

[7] Throughout this article the metric $g_{(0)}$ is assumed to be non-degenerate. For studies of the AdS/CFT correspondence in cases where $g_{(0)}$ is degenerate we refer to [9, 43].

[8] Greek indices, $\mu, \nu, ..$ are used for $d+1$-dimensional indices, Latin ones, $i, j, ..$ for d-dimensional ones. To distinguish the curvatures of the various metrics introduced in (2.3) we will often use the notation $R_{ij}[g]$ to indicate that this is the Ricci tensor of the metric g, etc.

One can check that the curvature of G satisfies

$$R_{\kappa\lambda\mu\nu}[G] = (G_{\kappa\mu}G_{\lambda\nu} - G_{\kappa\nu}G_{\lambda\mu}) + \mathcal{O}(\rho) \tag{2.4}$$

In this sense the metric is asymptotically anti-de Sitter. The Dirichlet problem for Einstein metrics satisfying (2.4) exactly (i.e. not only to leading order in ρ) was solved in [40].

In the coordinate system (2.3), Einstein's equations read [26]

$$\begin{aligned}
\rho\,[2g'' - 2g'g^{-1}g' + \mathrm{Tr}\,(g^{-1}g')\,g'] + \mathrm{Ric}(g) - (d-2)\,g' - \mathrm{Tr}\,(g^{-1}g')\,g &= 0\\
\nabla_i\,\mathrm{Tr}\,(g^{-1}g') - \nabla^j g'_{ij} &= 0\\
\mathrm{Tr}\,(g^{-1}g'') - \frac{1}{2}\mathrm{Tr}\,(g^{-1}g'g^{-1}g') &= 0,
\end{aligned} \tag{2.5}$$

where differentiation with respect to ρ is denoted with a prime, ∇_i is the covariant derivative constructed from the metric g, and $\mathrm{Ric}(g)$ is the Ricci tensor of g.

These equations are solved order by order in ρ. This is achieved by differentiating the equations with respect to ρ and then setting $\rho = 0$. For even d, this process would have broken down at order $d/2$ if the logarithm was not introduced in (2.3). $h_{(d)}$ is traceless, $\mathrm{Tr}\, g_{(0)}^{-1} h_{(d)} = 0$, and covariantly conserved, $\nabla^i h_{(d)ij} = 0$. We show in appendix C that $h_{(d)}$ is proportional to the metric variation of the corresponding conformal anomaly, i.e. it is proportional to the stress-energy tensor of the theory with action the conformal anomaly. In any dimension, only the trace of $g_{(d)}$ and its covariant divergence are determined. Here is where extra data from the CFT are needed: as we shall see, the undetermined part is specified by the expectation value of the dual stress-energy tensor.

We collect in appendix A the results for $g_{(n)}$, $h_{(d)}$ as well as the results for the trace and divergence $g_{(d)}$. In dimension d the latter are the only constraints that equations (2.5) yield for $g_{(d)}$. From this information we can parametrize the indeterminacy by finding the most general $g_{(d)}$ that has the determined trace and divergence.

In $d = 2$ and $d = 4$ the equation for the coefficient $g_{(d)}$ has the form of a conservation law

$$\nabla^i g_{(d)ij} = \nabla^i A_{(d)ij} \quad , \qquad d = 2, 4 \tag{2.6}$$

where $A_{(d)ij}$ is a symmetric tensor explicitly constructed from the coefficients $g_{(n)}$, $n < d$. The precise form of the tensor $A_{(d)ij}$ is given in appendix A (eq.(A.4)). The integration of this equation obviously involves an "integration constant" $t_{ij}(x)$, a symmetric covariantly conserved tensor the precise form of which can not be determined from Einstein's equations.

In two dimensions, we get [40] (see also [6])

$$g_{(2)ij} = \frac{1}{2}(R\, g_{(0)ij} + t_{ij}), \tag{2.7}$$

where the symmetric tensor t_{ij} should satisfy

$$\nabla^i t_{ij} = 0, \qquad \mathrm{Tr}\, t = -R. \tag{2.8}$$

In four dimensions we obtain[9]

$$g_{(4)ij} = \frac{1}{8} g_{(0)ij}\,[(\mathrm{Tr}\, g_{(2)})^2 - \mathrm{Tr}\, g_{(2)}^2] + \frac{1}{2}(g_{(2)}^2)_{ij} - \frac{1}{4} g_{(2)ij}\,\mathrm{Tr}\, g_{(2)} + t_{ij}, \tag{2.9}$$

[9]From now on we will suppress factors of $g_{(0)}$. For instance, $\mathrm{Tr}\, g_{(2)} g_{(4)} = \mathrm{Tr}\,[g_{(0)}^{-1} g_{(2)} g_{(0)}^{-1} g_{(4)}]$. Unless we explicitly mention to the contrary, indices will be raised and lowered with the metric $g_{(0)}$, all contractions will be made with this metric.

The tensor t_{ij} satisfies

$$\nabla^i t_{ij} = 0, \qquad \mathrm{Tr}\, t = -\frac{1}{4}[(\mathrm{Tr}\, g_{(2)})^2 - \mathrm{Tr}\, g_{(2)}^2]. \tag{2.10}$$

In six dimensions the equation determining the coefficient $g_{(6)}$ is more subtle than the one in (2.6). It given by

$$\nabla^i g_{(6)ij} = \nabla^i A_{(6)ij} + \frac{1}{6}\mathrm{Tr}(g_{(4)}\nabla_j g_{(2)}) \tag{2.11}$$

where the tensor $A_{(6)ij}$ is given in (A.4). It contains a part which is antisymmetric in the indices i and j. Since $g_{(6)ij}$ is by definition a symmetric tensor the integration of equation (2.11) is not straightforward. Moreover, it is not obvious that the last term in (2.11) takes a form of divergence of some local tensor. Nevertheless, this is indeed the case as we now show. Let us define the tensor S_{ij},

$$\begin{aligned} S_{ij} &= \nabla^2 C_{ij} - 2R^{k\ \ l}_{\ i\ j} C_{kl} + 4(g_{(2)}g_{(4)} - g_{(4)}g_{(2)})_{ij} + \frac{1}{10}(\nabla_i\nabla_j B - g_{(0)ij}\nabla^2 B) \\ &+ \frac{2}{5}g_{(2)ij}B + g_{(0)ij}(-\frac{2}{3}\mathrm{Tr}\, g_{(2)}^3 - \frac{4}{15}(\mathrm{Tr}\, g_{(2)})^3 + \frac{3}{5}\mathrm{Tr}\, g_{(2)}\mathrm{Tr}\, g_{(2)}^2) \quad , \end{aligned} \tag{2.12}$$

where

$$C_{ij} = (g_{(4)} - \frac{1}{2}g_{(2)}^2 + \frac{1}{4}g_{(2)}\mathrm{Tr}\, g_{(2)})_{ij} + \frac{1}{8}g_{(0)ij}B \quad , \quad B = \mathrm{Tr}\, g_2^2 - (\mathrm{Tr}\, g_2)^2 \quad .$$

The tensor S_{ij} is a local function of the Riemann tensor. Its divergence and trace read

$$\nabla^i S_{ij} = -4\mathrm{Tr}(g_{(4)}\nabla_j g_{(2)}) \quad , \quad \mathrm{Tr}S = -8\mathrm{Tr}(g_{(2)}g_{(4)}) \quad . \tag{2.13}$$

With the help of the tensor S_{ij} the equation (2.11) can be integrated in a way similar to the $d = 2, 4$ cases. One obtains

$$g_{(6)ij} = A_{(6)ij} - \frac{1}{24}S_{ij} + t_{ij} \quad . \tag{2.14}$$

Notice that tensor S_{ij} contains an antisymmetric part which cancels the antisymmetric part of the tensor $A_{(6)ij}$ so that $g_{(6)ij}$ and t_{ij} are symmetric tensors, as they should. The symmetric tensor t_{ij} satisfies

$$\nabla^i t_{ij} = 0 \quad , \quad \mathrm{Tr}\, t = -\frac{1}{3}[\frac{1}{8}(\mathrm{Tr} g_{(2)})^3 - \frac{3}{8}\mathrm{Tr} g_{(2)}\mathrm{Tr} g_{(2)}^2 + \frac{1}{2}\mathrm{Tr} g_{(2)}^3 - \mathrm{Tr} g_{(2)}g_{(4)}] \quad . \tag{2.15}$$

Notice that in all three cases, $d = 2, 4, 6$, the trace of t_{ij} is proportional to the holographic conformal anomaly. As we will see in the next section, the symmetric tensors t_{ij} are directly related to the expectation value of the boundary stress-energy tensor.

When d is odd the only constraint on the coefficient $g_{(d)ij}(x)$ is that it is conserved and traceless

$$\nabla^i g_{(d)ij} = 0 \quad , \qquad \mathrm{Tr}\, g_{(d)} = 0 \quad . \tag{2.16}$$

So that we may identify

$$g_{(d)ij} = t_{ij} \quad . \tag{2.17}$$

3 The holographic stress-energy tensor

We have seen in the previous section that given a conformal structure at infinity we can determine an asymptotic expansion of the metric up to order $\rho^{d/2}$. We will now show that this term is determined by the expectation value of the dual stress-energy tensor.

According to the AdS/CFT prescription, the expectation value of the boundary stress-energy tensor is determined by functionally differentiating the on-shell gravitational action with respect to the boundary metric. The on-shell gravitational action, however, diverges. To regulate the theory we restrict the bulk integral to the region $\rho \geq \epsilon$ and we evaluate the boundary term at $\rho = \epsilon$. The regulated action is given by

$$\begin{aligned} S_{\rm gr,reg} &= \frac{1}{16\pi G_{\rm N}} \left[\int_{\rho \geq \epsilon} {\rm d}^{d+1}x \sqrt{G} (R[G] + 2\Lambda) - \int_{\rho=\epsilon} {\rm d}^d x \sqrt{\gamma}\, 2K \right] = \\ &= \frac{1}{16\pi G_{\rm N}} \int {\rm d}^d x \left[\int_\epsilon {\rm d}\rho \, \frac{d}{\rho^{d/2+1}} \sqrt{\det g(x,\rho)} + \frac{1}{\rho^{d/2}} (-2d\sqrt{\det g(x,\rho)} + 4\rho \partial_\rho \sqrt{\det g(x,\rho)})|_{\rho=\epsilon} \right] \end{aligned} \tag{3.1}$$

Evaluating (3.1) for the solution we obtained in the previous section we find that the divergences appears as $1/\epsilon^k$ poles plus a logarithmic divergence [26],

$$S_{\rm gr,reg} = \frac{l}{16\pi G_{\rm N}} \int {\rm d}^d x \sqrt{\det g_{(0)}} \left(\epsilon^{-d/2} a_{(0)} + \epsilon^{-d/2+1} a_{(2)} + \ldots + \epsilon^{-1} a_{(d-2)} - \log \epsilon \, a_{(d)} \right) + \mathcal{O}(\epsilon^0), \tag{3.2}$$

where the coefficients $a_{(n)}$ are local covariant expressions of the metric $g_{(0)}$ and its curvature tensor. We give the explicit expressions, up to the order we are interested in, in appendix B.

We now obtain the renormalized action by subtracting the divergent terms, and then removing the regulator,

$$S_{\rm gr,ren}[g_{(0)}] = \lim_{\epsilon \to 0} \frac{1}{16\pi G_{\rm N}} [S_{\rm gr,reg} - \int {\rm d}^d x \sqrt{\det g_{(0)}} \left(\epsilon^{-d/2} a_{(0)} + \epsilon^{-d/2+1} a_{(2)} + \ldots + \epsilon^{-1} a_{(d-2)} - \log \epsilon \, a_{(d)} \right)] \tag{3.3}$$

The expectation value of the stress-energy tensor of the dual theory is given by

$$\langle T_{ij} \rangle = \frac{2}{\sqrt{\det g_{(0)}}} \frac{\partial S_{\rm gr,ren}}{\partial g_{(0)}^{ij}} = \lim_{\epsilon \to 0} \frac{2}{\sqrt{\det g(x,\epsilon)}} \frac{\partial S_{\rm gr,ren}}{\partial g^{ij}(x,\epsilon)} = \lim_{\epsilon \to 0} \left(\frac{1}{\epsilon^{d/2-1}} T_{ij}[\gamma] \right) \tag{3.4}$$

where $T_{ij}[\gamma]$ is the stress-energy tensor of the theory at $\rho = \epsilon$ described by the action in (3.3) but before the limit $\epsilon \to 0$ is taken ($\gamma_{ij} = 1/\epsilon \, g_{ij}(x,\epsilon)$ is the induced metric at $\rho = \epsilon$). Notice that the asymptotic expansion of the metric only allows for the determination of the divergences of the on-shell action. We can still obtain, however, a formula for $\langle T_{ij} \rangle$ in terms of $g_{(n)}$ since, as (3.4) shows, we only need to know the first $\epsilon^{d/2-1}$ orders in the expansion of $T_{ij}[\gamma]$.

The stress-energy tensor $T_{ij}[\gamma]$ contains two contributions,

$$T_{ij}[\gamma] = T_{ij}^{\rm reg} + T_{ij}^{\rm ct}, \tag{3.5}$$

$T_{ij}^{\rm reg}$ comes from the regulated action in (3.1) and $T_{ij}^{\rm ct}$ is due to the counterterms. The first contribution is equal to

$$T_{ij}^{\rm reg}[\gamma] = -\frac{1}{8\pi G_{\rm N}} (K_{ij} - K\gamma_{ij}) = -\frac{1}{8\pi G_{\rm N}} \left(-\partial_\epsilon g_{ij}(x,\epsilon) + g_{ij}(x,\epsilon) \, {\rm Tr}[g^{-1}(x,\epsilon) \partial_\epsilon g(x,\epsilon)] + \frac{1-d}{\epsilon} g_{ij}(x,\epsilon) \right) \tag{3.6}$$

The contribution due to counterterms can be obtained from the results in appendix B. It is given by

$$\begin{aligned} T_{ij}^{\rm ct} = & -\frac{1}{8\pi G_{\rm N}} \Bigg((d-1)\gamma_{ij} + \frac{1}{(d-2)} (R_{ij} - \frac{1}{2} R \gamma_{ij}) \\ & - \frac{1}{(d-4)(d-2)^2} [-\nabla^2 R_{ij} + 2 R_{ikjl} R^{kl} + \frac{d-2}{2(d-1)} \nabla_i \nabla_j R - \frac{d}{2(d-1)} R R_{ij} \\ & - \frac{1}{2} \gamma_{ij} (R_{kl} R^{kl} - \frac{d}{4(d-1)} R^2 - \frac{1}{d-1} \nabla^2 R)] - T_{ij}^a \log \epsilon \Bigg) \end{aligned} \tag{3.7}$$

where T^a_{ij} is the stress-energy tensor of the action $\int d^d x \sqrt{\det \gamma}\, a_{(d)}$. As it is shown in Appendix C, T^a_{ij} is proportional to the tensor $h_{(d)ij}$ appearing in the expansion (2.3).

The stress tensor $T_{ij}[g_{(0)}]$ is covariantly conserved with respect to the metric $g_{(0)ij}$. To see this, notice that each of $T^{\rm reg}_{ij}$ and $T^{\rm ct}_{ij}$ is separately covariantly conserved with respect to the induced metric γ_{ij} at $\rho = \epsilon$: for $T^{\rm reg}_{ij}$ one can check this by using the second equation in (2.5), for $T^{\rm ct}_{ij}$ this follows from the fact that it was obtained by varying a local covariant counterterm. Since all divergences cancel in (3.4), we obtain that the finite part in (3.4) is conserved with respect to the metric $g_{(0)ij}$.

We are now ready to calculate T_{ij}. By construction (and we will verify this below) the divergent pieces cancel between $T^{\rm reg}$ and $T^{\rm ct}$.

3.1 $d = 2$

In two dimensions we obtain

$$\langle T_{ij} \rangle = \frac{l}{16\pi G_{\rm N}}\, t_{ij} \tag{3.8}$$

where we have used (2.7) and (2.8) and the fact that $T^a_{ij} = 0$ since $\int R$ is a topological invariant (and reinstated the factor of l). As promised, t_{ij} is directly related to the boundary stress-energy tensor. Taking the trace we obtain

$$\langle T^i_i \rangle = -\frac{c}{24\pi}\, R \tag{3.9}$$

where $c = 3l/2G_{\rm N}$, which is the correct conformal anomaly [10].

Using our results, one can immediately obtain the stress-energy tensor of the boundary theory associated with a given solution G of the three dimensional Einstein equations: one needs to write the metric in the coordinate system (2.3) and then use the formula

$$\langle T_{ij} \rangle = \frac{2l}{16\pi G_{\rm N}}\,(g_{(2)ij} - g_{(0)ij}\,{\rm Tr}\, g_{(2)}). \tag{3.10}$$

From the gravitational point of view this is the quasi-local stress energy tensor associated with the solution G.

3.2 $d = 4$

To obtain T_{ij} we first need to rewrite the expressions in $T^{\rm ct}$ in terms of $g_{(0)}$. This can be done with the help of the relation

$$R_{ij}[\gamma] = R_{ij}[g_{(0)}] + \frac{1}{4}\,\epsilon \left(2R_{ik}R^k{}_j - 2R_{ikjl}R^{kl} - \frac{1}{3}\nabla_i\nabla_j R + \nabla^2 R_{ij} - \frac{1}{6}\nabla^2 R g_{(0)ij}\right) + \mathcal{O}(\epsilon^2). \tag{3.11}$$

After some algebra one obtains,

$$\begin{aligned}\langle T_{ij}[g_{(0)}]\rangle = -\frac{1}{8\pi G_{\rm N}} \lim_{\epsilon\to 0} \quad & \Big[\frac{1}{\epsilon}(-g_{(2)ij} + g_{(0)ij}{\rm Tr}\, g_{(2)} + \frac{1}{2}R_{ij} - \frac{1}{4}g_{(0)ij}R) \\ & + \log\epsilon\,(-2h_{(4)ij} - T^a_{ij}) \\ & -2g_{(4)ij} - h_{(4)ij} - g_{(2)ij}{\rm Tr}\, g_{(2)} - \frac{1}{2}g_{(0)ij}{\rm Tr}\, g^2_{(2)} \\ & \frac{1}{8}(R_{ik}R^k{}_j - 2R_{ikjl}R^{kl} - \frac{1}{3}\nabla_i\nabla_j R + \nabla^2 R_{ij} - \frac{1}{6}\nabla^2 R g_{(0)ij})\end{aligned}$$

$$-\frac{1}{4}g_{(2)ij}R + \frac{1}{8}g_{(0)ij}(R_{kl}R^{kl} - \frac{1}{6}R^2)\Big]\,. \tag{3.12}$$

Using the explicit expression for $g_{(2)}$ and $h_{(4)}$ given in (A.1) and (A.6) one finds that both the $1/\epsilon$ pole and the logarithmic divergence cancel. Notice that had we not subtracted the logarithmic divergence from the action, the resulting stress-energy tensor would have been singular in the limit $\epsilon \to 0$.

Using (2.9) and (2.10) and after some algebra we obtain

$$\langle T_{ij}\rangle = -\frac{1}{8\pi G_{\rm N}}[-2t_{ij} - 3h_{(4)}]. \tag{3.13}$$

Taking the trace we get

$$\langle T^i_i\rangle = \frac{1}{16\pi G_{\rm N}}(-2a_{(4)}), \tag{3.14}$$

which is the correct conformal anomaly [26].

Notice that since $h_{(4)ij} = -\frac{1}{2}T^a_{ij}$ the contribution in the boundary stress energy tensor proportional to $h_{(4)ij}$ is scheme dependent. Adding a local finite counterterm proportional to the trace anomaly will change the coefficient of this term. One may remove this contribution from the boundary stress energy tensor by a choice of scheme.

Finally, one can obtain the energy-momentum tensor of the boundary theory for a given solution G of the five dimensional Einstein equations with negative cosmological constant. It is given by

$$\langle T_{ij}\rangle = \frac{4}{16\pi G_{\rm N}}[g_{(4)ij} - \frac{1}{8}g_{(0)ij}[(\mathrm{Tr}\, g_{(2)})^2 - \mathrm{Tr}\, g^2_{(2)}] - \frac{1}{2}(g^2_{(2)})_{ij} + \frac{1}{4}g_{(2)ij}\mathrm{Tr}\, g_{(2)}], \tag{3.15}$$

where we have omitted the scheme dependent $h_{(4)}$ terms. From the gravitational point of view this is the quasi-local stress energy tensor associated with the solution G.

3.3 $d = 6$

The calculation of the boundary stress tensor in $d = 6$ case goes along the same lines as in $d = 2$ and $d = 4$ cases although it is technically involved. Up to local traceless covariantly conserved term (proportional to $h_{(6)}$) the results is

$$\langle T_{ij}\rangle = \frac{3}{8\pi G_{\rm N}}\left(g_{(6)ij} - A_{(6)ij} + \frac{1}{24}S_{ij}\right)\quad . \tag{3.16}$$

where $A_{(6)ij}$ is given in (A.4) and S_{ij} in (2.12). It is covariantly conserved and has the correct trace

$$\langle T^i_i\rangle = \frac{1}{8\pi G_{\rm N}}(-a_{(6)})\quad , \tag{3.17}$$

reproducing correctly the conformal anomaly in six dimensions [26].

Given an asymptotically AdS solution in six dimensions equation (3.16) yields the quasi-local stress energy tensor associated with it.

3.4 $d = 2k + 1$

In this case one can check that the counterterms only cancel infinities. Evaluating the finite part we get

$$\langle T_{ij}\rangle = \frac{d}{16\pi G_{\rm N}}\, g_{(d)ij}. \tag{3.18}$$

where $g_{(d)ij}$ can be identified with a traceless covariantly conserved tensor t_{ij}. In odd boundary dimensions there are no gravitational conformal anomalies, and indeed (3.18) is traceless. As in all previous cases, one can also read (3.18) as giving the quasi-local stress energy tensor associated with a given solution of Einstein's equations.

3.5 Conformally flat bulk metrics

In this subsection we discuss a special case where the bulk metric can be determined to all orders given only a boundary metric. It was shown in [40] that, given a conformally flat boundary metric, equations (2.5) can be integrated to all orders if the bulk Weyl tensor vanishes[10]. We show that the extra condition in the bulk metric singles out a specific vacuum of the CFT.

The solution obtained in [40] is given by

$$g(x,\rho) = g_{(0)}(x) + g_{(2)}(x)\rho + g_{(4)}(x)\rho^2 \quad , \quad g_{(4)} = \frac{1}{4}(g_{(2)})^2 \tag{3.19}$$

where $g_{(2)}$ is given in (A.1) (we consider $d > 2$), and all other coefficients $g_{(n)}$, $n > 4$ vanish. Since $g_{(4)}$ and $g_{(6)}$ are now known, one can obtain a local formula for the dual stress energy tensor in terms of the curvature by using (2.9) and (2.14).

In $d = 4$, using (2.9) and $g_{(4)} = \frac{1}{4}(g_{(2)})^2$, one obtains

$$t_{ij} = t_{ij}^{\mathrm{cf}} \equiv -\frac{1}{4}(g_{(2)})^2_{ij} + \frac{1}{4}g_{(2)ij}\mathrm{Tr}\, g_{(2)} - \frac{1}{8}g_{(0)ij}[(\mathrm{Tr}\, g_{(2)})^2 - \mathrm{Tr}\, g_{(2)}^2] \quad . \tag{3.20}$$

It is easy to check that trace of t_{ij}^{cf} reproduces (2.10). Furthermore, by virtue of Bianchi's, one can show that t_{ij}^{cf} is covariantly conserved. It is well-known that the stress-energy tensor of a quantum field theory on a conformally flat spacetime is a local function of the curvature tensor (see for example [8]). Our equation (3.20) reproduces the corresponding expression given in [8].

In $d = 6$, using (2.14) and $g_{(6)} = 0$ we find

$$\begin{aligned} t_{ij} = t_{ij}^{\mathrm{cf}} \equiv [&\frac{1}{4}g_{(2)}^3 - \frac{1}{4}g_{(2)}^2\mathrm{Tr}\, g_{(2)} + \frac{1}{8}g_{(2)}(\mathrm{Tr} g_{(2)})^2 - \frac{1}{8}g_{(2)}\mathrm{Tr}\, g_{(2)} \\ &+ g_{(0)}(\frac{1}{8}\mathrm{Tr}\, g_{(2)}\mathrm{Tr}\, g_{(2)}^2 - \frac{1}{12}\mathrm{Tr}\, g_{(2)}^3 - \frac{1}{24}(\mathrm{Tr}\, g_{(2)})^3)]_{ij} \quad . \end{aligned} \tag{3.21}$$

One can verify that the trace of t_{ij}^{cf} reproduces (2.15) (taking into account that $g_{(4)} = \frac{1}{4}g_{(2)}^2$) and that t_{ij}^{cf} is covariantly conserved (by virtue of Bianchi's).

Following the analysis in the previous subsections we obtain

$$\langle T_{ij}\rangle = \frac{d}{16\pi G_{\mathrm{N}}}\, t_{ij}^{\mathrm{cf}}. \tag{3.22}$$

So, we explicitly see that the global condition we imposed on the bulk metric implies that we have picked a particular vacuum in the conformal field theory.

Note that the tensors t_{ij}^{cf} in (3.20), (3.21) are local polynomial functions of the Ricci scalar and the Ricci tensor (but not of the Riemann tensor) of the metric $g_{(0)ij}$. It is perhaps an expected but still a surprising result that in conformally flat backgrounds the anomalous stress tensor is a local function of the curvature.

[10] In [40] it was proven that if the bulk metric satisfies Einstein's equations and it has a vanishing Weyl tensor, then the corresponding boundary metric has to be conformally flat. The converse is not necessarily true: one can have Einstein metrics with non-vanishing Weyl tensor which induce a conformally flat metric in the boundary.

4 Conformal transformation properties of the stress-energy tensor

In this section we discuss the conformal transformation properties of the stress-energy tensor. These can be obtained by noting [28] that conformal transformations in the boundary originate from specific diffeomorphisms that preserve the form of the metric (2.3). Under these diffeomorphisms $g_{ij}(x,\rho)$ transforms infinitesimally as [28]

$$\delta g_{ij}(x,\rho) = 2\sigma(1-\rho\partial_\rho)\, g_{ij}(x,\rho) + \nabla_i a_j(x,\rho) + \nabla_j a_i(x,\rho), \tag{4.1}$$

where $a_j(x,\rho)$ is obtained from the equation

$$a^i(x,\rho) = \frac{1}{2}\int_0^\rho \mathrm{d}\rho' g^{ij}(x,\rho')\partial_j \sigma(x). \tag{4.2}$$

This can be integrated perturbatively in ρ,

$$a^i(x,\rho) = \sum_{k=1} a^i_{(k)}\rho^k. \tag{4.3}$$

We will need the first two terms in this expansion,

$$a^i_{(1)} = \frac{1}{2}\partial^i\sigma, \qquad a^i_{(2)} = -\frac{1}{4}g^{ij}_{(2)}\partial_j\sigma. \tag{4.4}$$

We can now obtain the way the $g_{(n)}$'s transform under conformal transformations [28]

$$\begin{aligned}
\delta g_{(0)ij} &= 2\sigma g_{(0)ij}, \\
\delta g_{(2)ij} &= \nabla_i a_{(1)j} + \nabla_j a_{(1)i} \\
\delta g_{(3)ij} &= -\sigma g_{(3)ij}, \\
\delta g_{(4)ij} &= -2\sigma(g_{(4)} + h_{(4)}) + a^k_{(1)}\nabla_k g_{(2)ij} + \nabla_i a_{(2)j} + \nabla_j a_{(2)i} + g_{(2)ik}\nabla_j a^k_{(1)} + g_{(2)jk}\nabla_i a^k_{(1)} \\
\delta g_{(5)ij} &= -3\sigma g_{(3)ij},
\end{aligned} \tag{4.5}$$

where the term $h_{(4)}$ in $g_{(4)}$ is only present when $d=4$. One can check from the explicit expressions for $g_{(2)}$ and $g_{(4)}$ in (A.1) that they indeed transform as (4.5). An alternative way to derive the transformation rules above is to start from (A.1) and perform a conformal variation. In [28] the variations (4.5) were integrated leading to (A.1) up to conformally invariant terms.

Equipped with these results and the explicit form of the energy-momentum tensors, we can now easily calculate how the quantum stress-energy tensor transforms under conformal transformations. We use the term "quantum stress-energy tensor" because it incorporates the conformal anomaly. In the literature such transformation rules were obtained [12] by first integrating the conformal anomaly to an effective action. This effective action is a functional of the initial metric g and of the conformal factor σ. It can be shown that the difference between the stress-energy tensor of the theory on the manifold with metric $ge^{2\sigma}$ and the one on the manifold with metric g is given by the stress-energy tensor derived by varying the effective action with respect to g.

In any dimension the stress-energy tensor transforms *classically* under conformal transformations as

$$\delta\langle T_{\mu\nu}\rangle = -(d-2)\,\sigma\,\langle T_{\mu\nu}\rangle \tag{4.6}$$

This transformation law is modified by the quantum conformal anomaly. In odd dimensions, where there is no conformal anomaly, the classical transformation rule (4.6) holds also at the quantum level. Indeed, for odd d, and by using (3.18) and (4.5), one easily verifies that the holographic stress-energy tensor transforms correctly.

In even dimensions, the transformation (4.6) is modified. In $d = 2$, it is well-known that one gets an extra contribution proportional to the central charge. Indeed, using (3.10) and the formulae above we obtain

$$\delta\langle T_{ij}\rangle = \frac{l}{8\pi G_{\rm N}}\,(\nabla_i\nabla_j\sigma - g_{(0)ij}\nabla^2\sigma) = \frac{c}{12}\,(\nabla_i\nabla_j\sigma - g_{(0)ij}\nabla^2\sigma), \tag{4.7}$$

which is the correct transformation rule.

In $d = 4$ we obtain,

$$\begin{aligned}\delta\langle T_{ij}\rangle \;=\; & -2\sigma\langle T_{ij}\rangle + \frac{1}{4\pi G_{\rm N}}\Big(-2\sigma h_{(4)} + \frac{1}{4}\nabla^k\sigma[\nabla_k R_{ij} - \frac{1}{2}(\nabla_i R_{jk} + \nabla_j R_{ik}) - \frac{1}{6}\nabla_k R g_{(0)ij}] \\ & + \frac{1}{48}(\nabla_i\sigma\nabla_j R + \nabla_i\sigma\nabla_j R) + \frac{1}{12}R(\nabla_i\nabla_j\sigma - g_{(0)ij}\nabla^2\sigma) \\ & + \frac{1}{8}[R_{ij}\nabla^2\sigma - (R_{ik}\nabla^k\nabla_j\sigma + R_{jk}\nabla^k\nabla_i\sigma) + g_{(0)ij}R_{kl}\nabla^k\nabla^l\sigma]\Big)\,. \end{aligned} \tag{4.8}$$

The only other result known to us is the result in [12], where they computed the finite conformal transformation of the stress-energy tensor but for a conformally flat metric $g_{(0)}$. For conformally flat backgrounds, $h_{(4)}$ vanishes because it is the metric variation of a topological invariant. The terms proportional to a single derivative of σ vanish by virtue of Bianchi identities and the fact that the Weyl tensor vanishes for conformally flat metrics. The remaining terms, which only contain second derivatives of σ, can be shown to coincide with the infinitesimal version of (4.23) in [12].

One can obtain the conformal transformation of the stress energy tensor in $d = 6$ in a similar fashion but we shall not present this result here.

5 Matter

In the previous sections we examined how spacetime is reconstructed (to leading order) holographically out of CFT data. In this section we wish to examine how field theory describing matter on this spacetime is encoded in the CFT. We will discuss scalar fields but the techniques are readily applicable to other kinds of matter.

The method we will use is the same as in the case of pure gravity, i.e. we will start by specifying the sources that are turned on, find how far we can go with only this information and then input more CFT data. We will find the same pattern: knowledge of the sources allows only for determination of the divergent part of the action. The leading finite part (which depends on global issues and/or the signature of spacetime) is determined by the expectation value of the dual operator. We would like to stress that in the approach we follow, i.e. regularize, subtract all infinities by adding counterterms and finally remove the regulator to obtain the renormalized action, all normalizations of the physical correlation functions are fixed and are consistent with Ward identities.

Other papers that discuss similar issues include [1, 36, 35, 44].

5.1 Dirichlet boundary problem for scalar fields in a fixed gravitational background

In this section we consider scalars on a fixed gravitational background. This is taken to be of the generic form (2.3). In most of the literature the fixed metric was taken to be that of standard AdS, but with not much more effort one can consider the general case.

The action for massive scalar is given by

$$S_{\mathrm{M}} = \frac{1}{2} \int \mathrm{d}^{d+1}x \sqrt{G} \left(G^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + m^2 \Phi^2 \right) \tag{5.1}$$

where $G_{\mu\nu}$ has an expansion of the form (2.3).

We take the scalar field Φ to have an expansion of the form

$$\Phi(x,\rho) = \rho^{(d-\Delta)/2} \phi(x,\rho), \qquad \phi(x,\rho) = \phi_{(0)} + \phi_{(2)}\rho + \ldots, \tag{5.2}$$

where Δ is the conformal dimension of the dual operator. We take the dimension Δ to be quantized as $\Delta = \frac{d}{2} + k$, $k = 0, 1, \ldots$ This is often the case for operators of protected dimension. For the case of scalars that correspond to operators of dimensions $\frac{d}{2} - 1 \leq \Delta < \frac{d}{2}$ we refer to [29]. Inserting (5.2) in the field equation,

$$(-\Box_G + m^2)\Phi = 0, \tag{5.3}$$

where $\Box_G \Phi = \frac{1}{\sqrt{G}} \partial_\mu (\sqrt{G} G^{\mu\nu} \partial_\nu \Phi)$, we obtain that the mass m^2 and the conformal dimension Δ are related as $m^2 = (\Delta - d)\Delta$, and that ϕ satisfies

$$[-(d-\Delta)\partial_\rho \log g \, \phi + 2(2\Delta - d - 2)\partial_\rho \phi - \Box_g \phi] + \rho[-2\partial_\rho \log g \, \partial_\rho \phi - 4\partial_\rho^2 \phi] = 0. \tag{5.4}$$

Given $\phi_{(0)}$ one can determine recursively $\phi_{(n)}, n > 0$. This is achieved by differentiating (5.4) and setting ρ equal to zero. We give the result for the first couple of orders in appendix D. This process breaks down at order $\Delta - d/2$ (provided this is an integer, which we assume throughout this section) because the coefficient of $\phi_{(2\Delta-d)}$ (the field to be determined) becomes zero. This is exactly analogous to the situation encountered for even d in the gravitational sector. Exactly the same way as there, we introduce at this order a logarithmic term, i.e. the expansion of Φ now reads,

$$\Phi = \rho^{(d-\Delta)/2} \left(\phi_{(0)} + \rho\phi_{(2)} + \ldots \right) + \rho^{\Delta/2} \left(\phi_{(2\Delta-d)} + \log\rho \, \psi_{(2\Delta-d)} + \ldots \right). \tag{5.5}$$

The equation (5.4) now determines all terms up to $\phi_{(2\Delta-d-2)}$, the coefficient of the logarithmic term $\psi_{(2\Delta-d)}$, but leaves undetermined $\phi_{(2\Delta-d)}$. This is analogous to the situation discussed in section 2 where the term $g_{(d)}$ was undetermined. It is well known [4, 5, 29] that precisely at order $\rho^{\Delta/2}$ one finds the expectation value of the dual operator. We will review this argument below, and also derive the exact proportionality coefficient. Our result is in agreement with [29].

We proceed to regularize and then renormalize the theory. We regulate by integrating in the bulk from $\rho \geq \epsilon$,[11]

$$S_{\mathrm{M,reg}} = \frac{1}{2} \int_{\rho \geq \epsilon} \mathrm{d}^{d+1}x \sqrt{G} \left(G^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + m^2 \Phi^2 \right)$$

[11] This regularization for scalar fields in a fixed AdS background was considered in [33, 17]. In these papers the divergences were computed in momentum space, but no counterterms were added to cancel them. Addition of boundary counterterms to cancel infinities for scalar fields was considered in [13], and more recently in [29].

$$= \; -\int_{\rho=\epsilon} \mathrm{d}^d x \, \sqrt{g(x,\epsilon)}\epsilon^{-\Delta+d/2} \, [\frac{1}{2}(d-\Delta)\phi^2(x,\epsilon) + \epsilon\,\phi(x,\epsilon)\partial_\epsilon\phi(x,\epsilon)] \tag{5.6}$$

$$= \; \int \mathrm{d}^d x \, \sqrt{g_{(0)}} \, [\epsilon^{-\Delta+d/2} a^{\mathrm{M}}_{(0)} + \epsilon^{-\Delta+d/2+1} a^{\mathrm{M}}_{(2)} + ... + \epsilon \, a^{\mathrm{M}}_{(2\Delta-d+2)} - \log\epsilon \, a_{(2\Delta-d)}] + \mathcal{O}(\epsilon^0)$$

Clearly, with $\Delta - d/2$ a positive integer there are finite number of divergent terms. The logarithmic divergence appears exactly when $\Delta = d/2 + k, k = 0, 1, ..,$ in agreement with the analysis in [37], and is directly related to the logarithmic term in (5.5). The first few of the power law divergences read

$$a^{\mathrm{M}}_{(0)} = -\frac{1}{2}(d-\Delta)\phi^2_{(0)}, \qquad a^{\mathrm{M}}_{(2)} = -\frac{1}{4}\mathrm{Tr}\, g_{(2)}\,\phi^2_{(0)} + (d-\Delta+1)\,\phi_{(0)}\phi_{(2)}. \tag{5.7}$$

Given a field of specific dimension it is straightforward to compute all divergent terms.

We now proceed to obtain the renormalized action by adding counterterms to cancel the infinities,

$$S_{\mathrm{M,ren}} = \lim_{\epsilon\to 0}[S_{\mathrm{M,reg}} - \int \mathrm{d}^d x \, \sqrt{g_{(0)}} \, [\epsilon^{-\Delta+d/2} a^{\mathrm{M}}_{(0)} + \epsilon^{-\Delta+d/2+1} a^{\mathrm{M}}_{(2)} + ... + \epsilon \, a^{\mathrm{M}}_{(2\Delta-d+2)} - \log\epsilon \, a_{(2\Delta-d)}] \tag{5.8}$$

Exactly as in the case of pure gravity, and since the regulated theory lives at $\rho = \epsilon$, one needs to rewrite the counterterms in terms of the field living at $\rho = \epsilon$, i.e. the induced metric $\gamma_{ij}(x,\epsilon)$ and the field $\Phi(x,\epsilon)$, or equivalently $g_{ij}(x,\epsilon)$ and $\phi(x,\epsilon)$. This is straightforward but somewhat tedious: one needs to invert the relation between ϕ and $\phi_{(0)}$ and between g_{ij} and $g_{(0)ij}$ to sufficiently high order. This then allows to express all $\phi_{(n)}$, and therefore all $a^{\mathrm{M}}_{(n)}$, in terms of $\phi(x,\epsilon)$ and $g_{ij}(x,\epsilon)$ (the $\phi_{(n)}$'s are determined in terms of $\phi_{(0)}$ and $g_{(0)}$ by solving (5.4) iteratively). Explicitly, the first two orders read,

$$\begin{aligned} S_{\mathrm{M,ren}} \;=\; & \lim_{\epsilon\to 0}\left[\frac{1}{2}\int_{\rho\geq\epsilon} \mathrm{d}^{d+1}x\sqrt{G}\,(G^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + m^2\Phi^2)\right. \\ & \left. + \int_{\rho=\epsilon}\sqrt{\gamma}\,[\frac{(d-\Delta)}{2}\Phi^2(x,\epsilon) + \frac{1}{2(2\Delta-d-2)}\,(\Phi(x,\epsilon)\Box_\gamma\Phi(x,\epsilon) + \frac{d-\Delta}{2(d-1)}R[\gamma]\Phi^2(x,\epsilon)) + ...]\right] \end{aligned} \tag{5.9}$$

The addition of the first counterterm was discussed in [29]. The action (5.9) with only the counterterms written explicitly is finite for fields of $\Delta < d/2 + 2$. As remarked above, it is straightforward to obtain all counterterms needed in order to make the action finite for any field of any mass. These counterterms contain also logarithmic subtractions that lead to the conformal anomalies discussed in [37]. For instance, if $\Delta = \frac{1}{2}d + 1$, the coefficient $[2(2\Delta - d - 2)]^{-1}$ in (5.9) is replaced by $-\frac{1}{4}\log\epsilon$. An alternative way to derive the counterterms is to demand that the expectation value $\langle O\rangle$ is finite. This holds in the case of pure gravity too, i.e. the counterterms can also be derived by requiring finiteness of $\langle T_{\mu\nu}\rangle$ [3].

The expectation value of the dual operator is given by

$$\langle O(x)\rangle = -\frac{1}{\sqrt{\det g_{(0)}}}\frac{\delta S_{\mathrm{M,ren}}}{\delta\phi_{(0)}} = -\lim_{\epsilon\to 0}\frac{1}{\sqrt{\det g(x,\epsilon)}}\frac{\delta S_{\mathrm{M,ren}}}{\delta\phi(x,\epsilon)}. \tag{5.10}$$

Exactly as in the case of pure gravity, the expectation value receives a contribution both from the regulated part and from the counterterms. We obtain,

$$\langle O(x)\rangle = (2\Delta - d)\,\phi_{(2\Delta-d)} + F(\phi_{(n)}, \psi_{(2\Delta-d)}, g_{(m)}), \qquad n < 2\Delta - d \tag{5.11}$$

where we used that $\phi_{(2\Delta-d)}$ is linear in $\phi_{(0)}$ (notice that the action (5.1) does not include interactions). $F(\phi_{(n)}, \psi_{(2\Delta-d)}, g_{(m)})$ is a local function of $\phi_{(n)}$ with $n < 2\Delta - d$, $\psi_{(2\Delta-d)}$ and $g_{(m)}$. These terms are related to contact terms in correlation functions of O with itself and with the stress-energy tensor. Its exact form is straightforward but somewhat tedious to obtain (just use (5.9) and (5.10)).

As we have promised, we have shown that the coefficient $\phi_{(2\Delta-d)}$ is related with the expectation value of the dual CFT operator. In the case that the background geometry is the standard Euclidean AdS one can readily obtain $\phi_{(2\Delta-d)}$ from the unique solution of the scalar field equation with given Dirichlet boundary conditions. One finds that $\phi_{(2\Delta-d)}$ is proportional to (an integral involving) $\phi_{(0)}$. Therefore, $\phi_{(2\Delta-d)}$ carries information about the 2-point function. The factor $(\Delta - d/2)$ is crucial in order for the 2-point function to be normalized correctly [17]. We refer to [29] for a detailed discussion of this point.

We finish this section by calculating the conformal anomaly associated with the scalar fields and in the case the background is (locally) standard AdS (i.e. $g_{(n)} = 0$, for $0 < n < d$). Equation (5.4) simplifies and can be easily solved. One gets

$$\begin{aligned} \phi_{(2n)} &= \frac{1}{2n(2\Delta-d-2n)} \Box_0 \phi_{(2n-2)}, \\ \psi_{(2\Delta-d)} &= -\frac{1}{2(2\Delta-d)} \Box_0 \phi_{(2\Delta-d-2)} = -\frac{1}{2^{2k}\Gamma(k)\Gamma(k+1)} (\Box_0)^k \phi_{(0)}, \end{aligned} \tag{5.12}$$

where $k = \Delta - \frac{d}{2}$ and $\Box_0$ is the Laplacian of $g_{(0)}$. The regularized action written in terms of the fields at $\rho = \epsilon$ contains the following explicit logarithmic divergence,

$$S_{\text{M,reg}} = -\int_{\rho=\epsilon} \mathrm{d}^d x \sqrt{\gamma} \left[\log\epsilon\, (\Delta - \frac{d}{2})\, \phi(x,\epsilon)\, \psi_{(2\Delta-d)}(x,\epsilon) + \cdots\right], \tag{5.13}$$

where the dots indicate power law divergent and finite terms, $\psi_{(2\Delta-d)}(x,\epsilon)$ is given by (5.12) with $g_{(0)}$ replaced by γ and $\phi_{(0)}$ by $\phi(x,\epsilon)$. Using the same argument as in [26] we obtain the matter conformal anomaly,

$$\mathcal{A}_{\text{M}} = \frac{1}{2}\left(\frac{1}{2^{2k-2}(\Gamma(k))^2}\right) \phi_{(0)} (\Box_0)^k \phi_{(0)}. \tag{5.14}$$

This agrees exactly with the anomaly calculated in [37] (compare with formulae (10), (37) in [37]).

5.2 Scalars coupled to gravity

In the previous section we ignored the back-reaction of the scalars to the bulk geometry. The purpose of this section is to discuss this issue. The action is now the sum of (2.1) and (5.1),

$$S = S_{\text{gr}} + S_{\text{M}}. \tag{5.15}$$

The gravitational field equation in the presence of matter reads

$$R_{\mu\nu} - \frac{1}{2}(R + 2\Lambda) G_{\mu\nu} = -8\pi G_{\text{N}} T_{\mu\nu} \tag{5.16}$$

In the coordinate system (2.3) and with the scalar field having the expansion in (5.5), these equations read

$$\begin{aligned} \rho\,[2g''_{ij} - 2(g'g^{-1}g')_{ij} + \text{Tr}\,(g^{-1}g')\,g'_{ij}\,] &+ R_{ij}(g) - (d-2)\,g'_{ij} - \text{Tr}\,(g^{-1}g')\,g_{ij} = \\ &= -8\pi G_{\text{N}}\,\rho^{d-\Delta-1}\left[\frac{(\Delta-d)\Delta}{d-1}\,\phi^2\,g_{ij} + \rho\,\partial_i\phi\partial_j\phi\right], \\ \nabla_i \text{Tr}\,(g^{-1}g') - \nabla^j g'_{ij} &= -16\pi G_{\text{N}}\,\rho^{d-\Delta-1}\left[\frac{d-\Delta}{2}\,\phi\partial_i\phi + \rho\,\partial_\rho\phi\partial_i\phi\right], \\ \text{Tr}\,(g^{-1}g'') - \frac{1}{2}\text{Tr}\,(g^{-1}g'g^{-1}g') &= -16\pi G_{\text{N}}\,\rho^{d-\Delta-2}\left[\frac{d(\Delta-d)(\Delta-d+1)}{4(d-1)}\,\phi^2\right. \\ &+ \left.(d-\Delta)\,\rho\,\phi\partial_\rho\phi + \rho^2\,(\partial_\rho\phi)^2\,\right], \end{aligned} \tag{5.17}$$

If $\Delta > d$, the right-hand side diverges near the boundary whereas the left-hand side is finite. Operators with dimension $\Delta > d$ are irrelevant operators. Correlation functions of these operators have a very complicated singularity structure at coincident points. As remarked in [47], one can avoid such problems by considering the sources to be infinitesimal and to have disjoint support, so that these operators are never at coincident points. Requiring that the equations in (5.17) are satisfied to leading order in ρ yields

$$\phi_{(0)}^2 = 0, \tag{5.18}$$

which is indeed the prescription advocated in [47].

If $\Delta \leq d$, which means that we deal with marginal or relevant operators, one can perturbatively calculate the back-reaction of the scalars to the bulk metric. At which order the leading back-reaction appears depends on the mass of the field. For fields that correspond to operators of dimension $\Delta = d - k$ the leading back-reaction appears at order ρ^k, except when $k = 0$ (marginal operators), where the leading back-reaction is at order ρ.

Let us see how conformal anomalies arise in this context. The logarithmic divergences are coming from the regulated on-shell value of the bulk integral in (5.15). The latter reads

$$\begin{aligned} S_{\rm reg}(\text{bulk}) &= \int_{\rho\geq\epsilon} \mathrm{d}\rho\, \mathrm{d}^d x \sqrt{G}\, [\frac{d}{8\pi G_{\rm N}} - \frac{m^2}{d-1}\,\Phi^2] \\ &= \int_{\rho\geq\epsilon} \mathrm{d}\rho\, \mathrm{d}^d x\, \frac{1}{\rho} \sqrt{g(x,\rho)}\, [\frac{d}{16\pi G_{\rm N}}\,\rho^{-d/2} - \frac{m^2}{2(d-1)}\,\phi^2(x,\rho)\,\rho^{-k}] \end{aligned} \tag{5.19}$$

where $k = \Delta - d/2$. We see that gravitational conformal anomalies are expected when d is even and matter conformal anomalies when k is a positive integer, as it should.

In the presence of sources the expectation value of the boundary stress-energy tensor is not conserved but rather it satisfies a Ward identity that relates its covariant divergence to the expectation value of the operators that couple to the sources. To see this consider the generating functional

$$Z_{\rm CFT}[g_{(0)}, \phi_{(0)}] = \big\langle \exp \int \mathrm{d}^d x \sqrt{g_{(0)}}\, [\frac{1}{2}\, g_{(0)}^{ij} T_{ij} - \phi_{(0)} O] \big\rangle. \tag{5.20}$$

Invariance under infinitesimal diffeomorphisms,

$$\delta g_{(0)ij} = \nabla_i \xi_j + \nabla_j \xi_i, \tag{5.21}$$

yields the Ward identity,

$$\nabla^j \langle T_{ij} \rangle = \langle O \rangle\, \partial_i \phi_{(0)}. \tag{5.22}$$

As we have remark before, $\langle T_{ij} \rangle$ has a dual meaning [3], both as the expectation value of the dual stress-energy tensor and as the quasi-local stress-energy tensor of Brown and York. The Ward identity (5.22) has a natural explanation from the latter point in view as well. According to [11] the quasi-local stress-energy tensor is not conserved in the presence of matter but it satisfies

$$\nabla^j \langle T_{ij} \rangle = -\tau_{i\rho} \tag{5.23}$$

where $\tau_{i\rho}$ expresses the flow of matter energy-momentum through the boundary. Evidently, (5.22) is of the form (5.23).

Solving the coupled system of equations (5.17) and (5.4) is straightforward but somewhat tedious. The details differ from case to case. For illustrative purposes we present a sample calculation: we consider the case of two-dimensional massless scalar field ($d = \Delta = 2, k = 1$).

The equations to be solved are (5.4) and (5.17) with $d = \Delta = 2$ and the expansion of the metric and the scalar field are given by (2.3) and (5.5) (again with $d = \Delta = 2$), respectively. Equation (5.4) determines $\psi_{(2)}$,

$$\psi_{(2)} = -\frac{1}{4}\Box_0 \phi_{(0)}. \tag{5.24}$$

Equations (5.17) determine $h_{(2)}$, the trace of the $g_{(2)}$ and provide a relation between the divergence of $g_{(2)}$ and $\phi_{(2)}$,

$$\begin{aligned}
h_{(2)} &= -4\pi G_{\mathrm{N}} \left(\partial_i \phi_{(0)} \partial_j \phi_{(0)} - \frac{1}{2} g_{(0)ij} \left(\partial \phi_{(0)}\right)^2 \right), \\
\mathrm{Tr}\, g_{(2)} &= \frac{1}{2} R + 4\pi G_{\mathrm{N}} \left(\partial \phi_{(0)}\right)^2, \\
\nabla^i g_{(2)ij} &= \partial_i \mathrm{Tr}\, g_{(2)} + 16\pi G_{\mathrm{N}}\, \phi_{(2)} \partial_i \phi_{(0)}.
\end{aligned} \tag{5.25}$$

Notice that $g_{(2)}$ and $\phi_{(2)}$ are still undetermined and are related to the expectation values of the dual operators (3.4) and (5.11), respectively. Notice that $h_{(2)}$ is equal to the stress-energy tensor of a massless two-dimensional scalar.

Going back to (5.19), we see that the second term drops out (since $m^2 = 0$) and one can use the result already obtained in the gravitational sector,

$$\mathcal{A} = \frac{1}{16\pi G_{\mathrm{N}}}(-2a_{(2)}) = \frac{1}{16\pi G_{\mathrm{N}}}(-2\mathrm{Tr}\, g_{(2)}) = -\frac{1}{16\pi G_{\mathrm{N}}} R + \frac{1}{2}\phi_{(0)} \Box_0 \phi_{(0)} - \frac{1}{2}\nabla_i(\phi_{(0)} \nabla^i \phi_{(0)}), \tag{5.26}$$

which is the correct conformal anomaly [26, 37] (the last term can be removed by adding a covariant counterterm).

The renormalized boundary stress tensor reads

$$\langle T_{ij}(x) \rangle = \frac{1}{8\pi G_{\mathrm{N}}} \left(g_{(2)ij} + h_{(2)ij} - g_{(0)ij} \mathrm{Tr}\, g_{(2)} \right)(x) \tag{5.27}$$

Its trace gives correctly the conformal anomaly (5.26). On the other hand, taking the covariant derivative of (5.27) we get

$$\begin{aligned}
\nabla^j \langle T_{ij} \rangle &= \langle O(x) \rangle\, \partial_i \phi_0(x) \\
\langle O(x) \rangle &= 2(\phi_2(x) + \psi_2(x)).
\end{aligned} \tag{5.28}$$

in agreement with equations (5.22) and (5.11).

6 Conclusions

Most of the discussions in the literature on the AdS/CFT correspondence are concerned with obtaining conformal field theory correlation functions using supergravity. In this paper we started investigating the converse question: how can one obtain information about the bulk theory from CFT correlation functions? How does one decode the hologram?

Answering these questions in all generality, but within the context of the AdS/CFT duality, entails developing a precise dictionary between bulk and boundary physics. A prescription for relating bulk/boundary observables is already available [24, 47], and one would expect that it would allow us to reconstruct the bulk spacetime from the boundary CFT. The prescription of [24, 47], however, relates infinite quantities. One of the main results of this paper is the systematic development of a renormalized version of this prescription. Equipped with it, and with no other assumption (except that the CFT has an AdS dual), we then proceeded to reconstruct the bulk spacetime metric and bulk scalar fields to the first non-trivial order.

Our approach to the problem is to start from the boundary and try to build iteratively bulk solutions. Within this approach, the pattern we find is the following:
• Sources in the CFT determine an asymptotic expansion of the corresponding bulk field near the boundary to high enough order so that *all infrared divergences* of the bulk on-shell action can be computed. This then allows to obtain a renormalized on-shell action by adding boundary counterterms to cancel the infrared divergences.
• Bulk solutions can be extended one order further by using the 1-point function of the corresponding dual CFT operator.

In the case the bulk field is the metric, our results show that a conformal structure at infinity is not in general sufficient in order to obtain a bulk metric. The first additional information one needs is the expectation value of the boundary stress energy tensor.

As a by-product, we have obtained ready-to-use formulae for the Brown-York quasi-local stress-energy tensor for arbitrary solution of Einstein's equations with negative cosmological constant up to six dimensions. The six-dimensional result is particularly interesting because, via AdS/CFT, provides new information about the still mysterious $(2,0)$ theory. Furthermore, we have obtained the conformal transformation properties of the stress-energy tensors. These transformation rules incorporate the trace anomaly and provide a generalization to $d>2$ of the well-known Schwartzian derivative contribution in the conformal transformation rule of the stress-energy tensor in $d=2$.

Our discussion extends straightforwardly to the case of different matter. We expect that in all cases obstructions in extending the solution to the deep interior region will be resolved by additional CFT data (including data about non-local observables such as Wilson loops, Wilson surfaces etc.). An interesting case to study in this framework is point particles [14]. Reconstructing the trajectory of the bulk point particle out of CFT data will present a model of how holography works with time dependent processes. Furthermore, following [27], one could study the interplay between causality and holography. Another extension is to study renormalization group flows using the present formalism. This amounts to extending the discussion in section 5.2 by adding a potential for the scalars. Another application of our results is in the context of Randall-Sundrum (RS) scenarios [38]. Incorporating such a scenario in string theory, in the case the bulk space is AdS, may yield a connection with the AdS/CFT duality [46,48]. As advocated in [48], one may view the RS scenario as $4d$ gravity coupled to a cut-off CFT. The regulated theory in our discussion provides a dual description of a cut-off CFT. In this context, the counterterms are re-interpreted as providing the action for the bulk modes localized in the brane [39,23,19]. We see, for instance, that the counterterms in (5.9) can be re-interpreted as an action for a bulk scalar mode localized on the brane.

Note added

As this paper was being finalized, [7] appeared with some overlap with the results of section 2.

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Appendix

A Asymptotic solution of Einstein's equations

In this appendix we collect the results for the solution of the equations (2.5) up to the order we are interested in.

From the first equation in (2.5) one determines the coefficients $g_{(n)}$, $n \neq d$, in terms of $g_{(0)}$. For our purpose we only need $g_{(2)}$ and $g_{(4)}$. There are given by

$$\begin{aligned}
g_{(2)ij} &= \frac{1}{d-2}\left(R_{ij} - \frac{1}{2(d-1)} R\, g_{(0)ij}\right) \\
g_{(4)ij} &= \frac{1}{d-4}\left(-\frac{1}{8(d-1)} D_i D_j R + \frac{1}{4(d-2)} D_k D^k R_{ij}\right. \\
&\quad - \frac{1}{8(d-1)(d-2)} D_k D^k R g_{(0)ij} - \frac{1}{2(d-2)} R^{kl} R_{ikjl} \\
&\quad + \frac{d-4}{2(d-2)^2} R_i{}^k R_{kj} + \frac{1}{(d-1)(d-2)^2} R R_{ij} \\
&\quad \left. + \frac{1}{4(d-2)^2} R^{kl} R_{kl} g_{(0)ij} - \frac{3d}{16(d-1)^2(d-2)^2} R^2 g_{(0)ij}\right). \qquad \text{(A.1)}
\end{aligned}$$

The expressions for $g_{(n)}$ are singular when $n = d$. One can obtain the trace and the divergence of $g_{(n)}$ for any n from the last two equations in (2.5). Explicitly,

$$\begin{aligned}
&\mathrm{Tr}\, g_{(4)} = \frac{1}{4} \mathrm{Tr}\, g_{(2)}^2, \qquad \mathrm{Tr}\, g_{(6)} = \frac{2}{3} \mathrm{Tr}\, g_{(2)} g_{(4)} - \frac{1}{6} \mathrm{Tr}\, g_{(2)}^3 \\
&\mathrm{Tr}\, g_{(3)} = 0, \qquad \mathrm{Tr}\, g_{(5)} = 0, \qquad \text{(A.2)}
\end{aligned}$$

and

$$\begin{aligned}
&\nabla^i g_{(2)ij} = \nabla^i A_{(2)ij}, \qquad \nabla^i g_{(3)ij} = 0, \qquad \nabla^i g_{(4)ij} = \nabla^i A_{(4)ij} \\
&\nabla^i g_{(5)ij} = 0, \qquad \nabla^i g_{(6)ij} = \nabla^i A_{(6)ij} + \frac{1}{6} \mathrm{Tr}\,(g_{(4)} \nabla_j g_{(2)}), \qquad \text{(A.3)}
\end{aligned}$$

where

$$\begin{aligned}
A_{(2)ij} &= g_{(0)ij} \mathrm{Tr}\, g_{(2)} \qquad \text{(A.4)} \\
A_{(4)ij} &= -\frac{1}{8}[\mathrm{Tr}\, g_{(2)}^2 - (\mathrm{Tr}\, g_{(2)})^2]\, g_{(0)ij} + \frac{1}{2}(g_{(2)}^2)_{ij} - \frac{1}{4} g_{(2)ij}\, \mathrm{Tr}\, g_{(2)} \\
A_{(6)ij} &= \frac{1}{3}\left(2(g_{(2)}g_{(4)})_{ij} + (g_{(4)}g_{(2)})_{ij} - (g_{(2)}^3)_{ij} + \frac{1}{8}[\mathrm{Tr}\, g_{(2)}^2 - (\mathrm{Tr}\, g_{(2)})^2]\, g_{(2)ij}\right. \\
&- \left. \mathrm{Tr}\, g_{(2)}\, [g_{(4)ij} - \frac{1}{2}(g_{(2)}^2)_{ij}] - [\frac{1}{8}\mathrm{Tr}\, g_{(2)}^2 \mathrm{Tr}\, g_{(2)} - \frac{1}{24}(\mathrm{Tr}\, g_{(2)})^3 - \frac{1}{6}\mathrm{Tr}\, g_{(2)}^3 + \frac{1}{2}\mathrm{Tr}\,(g_{(2)}g_{(4)})]\, g_{(0)ij}\right).
\end{aligned}$$

For even $n = d$ the first equation in (2.5) determines the coefficients $h_{(d)}$. They are given by

$$\begin{aligned}
h_{(2)ij} &= 0 \qquad \text{(A.5)} \\
h_{(4)ij} &= \frac{1}{2} g_{(2)ij}^2 - \frac{1}{8} g_{(0)ij} \mathrm{Tr}\, g_{(2)}^2 + \frac{1}{8}(\nabla^k \nabla_i g_{(2)jk} + \nabla^k \nabla_j g_{(2)ik} - \nabla^2 g_{(2)ij} - \nabla_i \nabla_j \mathrm{Tr}\, g_{(2)}) \qquad \text{(A.6)} \\
&= \frac{1}{8} R_{ikjl} R^{kl} + \frac{1}{48} \nabla_i \nabla_j R - \frac{1}{16} \nabla^2 R_{ij} - \frac{1}{24} R R_{ij} + (\frac{1}{96} \nabla^2 R + \frac{1}{96} R^2 - \frac{1}{32} R_{kl} R^{kl}) g_{(0)ij} \\
h_{(6)ij} &= \frac{2}{3}(g_{(4)}g_{(2)} + g_{(2)}g_{(4)})_{ij} - \frac{1}{3} g_{(2)ij}^3 - \frac{1}{6} g_{(4)ij} \mathrm{Tr}\, g_{(2)} + \frac{1}{6} g_{(0)ij}(3\mathrm{Tr} g_{(6)} - 3\mathrm{Tr} g_{(2)} g_{(4)} + \mathrm{Tr} g_{(2)}^3)
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{12}[-\frac{1}{4}\nabla_i\nabla_j \mathrm{Tr} g_{(2)}^2 - \nabla^k\nabla_i g_{(4)jk} - \nabla^k\nabla_j g_{(4)ik} + \nabla^2 g_{(4)ij} \\
&+g_{(2)}^{kl}[\nabla_l\nabla_i g_{(2)jk} + \nabla_l\nabla_j g_{(2)ik} - \nabla_l\nabla_k g_{(2)ij}] \\
&+\frac{1}{2}\nabla^k \mathrm{Tr} g_{(2)}(\nabla_i g_{(2)jk} + \nabla_j g_{(2)ik} - \nabla_k g_{(2)ij}) \\
&+\frac{1}{2}\nabla_i g_{(2)kl}\nabla_j g_{(2)}^{kl} + \nabla_k g_{(2)il}\nabla^l g_{(2)j}{}^k - \nabla_k g_{(2)il}\nabla^k g_{(2)j}{}^l]. \qquad \text{(A.7)}
\end{aligned}
$$

B Divergences in terms of the induced metric

In this appendix we rewrite the divergent terms of the regularized action in terms of the induced metric at $\rho = \epsilon$. This is needed in order to derive the contribution of the counterterms to the stress energy tensor.

The coefficients $a_{(n)}$ of the divergent terms in the regulated action (3.2) are given by

$$
\begin{aligned}
&a_{(0)} = 2(1-d), && a_{(2)} = b_{(2)}(d)\,\mathrm{Tr}\, g_{(2)}, \\
&a_{(4)} = b_{(4)}(d)\,[(\mathrm{Tr}\, g_{(2)})^2 - \mathrm{Tr}\, g_{(2)}^2], && a_{(6)} = \left(\frac{1}{8}\mathrm{Tr}\, g_{(2)}^3 - \frac{3}{8}\mathrm{Tr}\, g_{(2)}\mathrm{Tr}\, g_{(2)}^2 + \frac{1}{2}\mathrm{Tr}\, g_{(2)}^3 - \mathrm{Tr}\, g_{(2)}g_{(4)}\right),
\end{aligned} \qquad \text{(B.1)}
$$

where $a_{(6)}$ is only valid in six dimensions and the numerical coefficients in $a_{(2)}$ and $a_{(4)}$ are given by

$$
b_{(2)}(d\neq 2) = -\frac{(d-4)(d-1)}{d-2}, \quad b_{(2)}(d=2) = 1, \quad b_{(4)}(d\neq 4) = \frac{-d^2+9d-16}{4(d-4)}, \quad b_{(4)}(d=4) = \frac{1}{2}. \qquad \text{(B.2)}
$$

Notice that the coefficients $a_{(n)}$ are proportional to the expression for the conformal anomaly (in terms of $g_{(n)}$) in dimension $d = n$ [26].

The counterterms can be rewritten in terms of the induced metric by inverting the relation between γ and $g_{(0)}$ perturbatively in ϵ. One finds

$$
\begin{aligned}
\sqrt{g_{(0)}} &= \epsilon^{d/2}\left(1 - \frac{1}{2}\epsilon\,\mathrm{Tr}\, g_{(0)}^{-1}g_{(2)} + \frac{1}{8}\epsilon^2\,[(\mathrm{Tr}\, g_{(0)}^{-1}g_{(2)})^2 + \mathrm{Tr}\,(g_{(0)}^{-1}g_{(2)})^2] + \mathcal{O}(\epsilon^3)\right)\sqrt{\gamma} \\
\mathrm{Tr}\, g_{(2)} &= \frac{1}{2(d-1)}\frac{1}{\epsilon}\left(R[\gamma] + \frac{1}{d-2}(R_{ij}[\gamma]R^{ij}[\gamma] - \frac{1}{2(d-1)}R^2[\gamma]) + \mathcal{O}(R[\gamma]^3)\right) \\
\mathrm{Tr}\, g_{(2)}^2 &= \frac{1}{\epsilon^2}\frac{1}{(d-2)^2}\left(R_{ij}[\gamma]R^{ij}[\gamma] + \frac{-3d+4}{4(d-1)^2}R^2[\gamma] + \mathcal{O}(R[\gamma]^3)\right)
\end{aligned} \qquad \text{(B.3)}
$$

The terms cubic in curvatures in (B.3) give vanishing contribution in (3.4) up to six dimensions.

Putting everything together we obtain that the counterterms, rewritten in terms of the induced metric, are given by

$$
S^{\mathrm{ct}} = -\frac{1}{16\pi G_{\mathrm{N}}}\int_{\rho=\epsilon}\sqrt{\gamma}\left[2(1-d) + \frac{1}{d-2}R - \frac{1}{(d-4)(d-2)^2}(R_{ij}R^{ij} - \frac{d}{4(d-1)}R^2) - \log\epsilon\, a_{(d)} + \ldots\right] \qquad \text{(B.4)}
$$

where all quantities are now in terms of the induced metric, including the one in the logarithmic divergence. These are exactly the counterterms in [3, 15, 30] except that these authors did not include the logarithmic divergence. Equation (B.4) should be understood as containing only divergent counterterms in each dimension. This means that in even dimension $d = 2k$ one should include only the first k counterterms and the logarithmic one. In odd $d = 2k+1$, only the first $k+1$ counterterms should be included.

The logarithmic counterterms appear only for d even. The counterterms in (B.4) render the renormalized action finite up to $d = 6$. This covers all cases relevant for the AdS/CFT correspondence. It is straightforward but tedious to compute the necessary counterterms for $d > 6$. From (B.4) one straightforwardly obtains (3.7).

C Relation between $h_{(d)}$ and the conformal anomaly $a_{(d)}$

We show in this appendix that the tensor $h_{(d)}$ appearing in expansion of the metric in (2.3) when d is even is a multiple of the stress tensor derived from the action $\int a_{(d)}$. ($a_{(d)}$ is, up to a constant, the holographic conformal anomaly).

This can be shown by deriving the stress-energy tensor of the regulated theory at $\rho = \epsilon$ in two ways and then comparing the results. In the first derivation one starts from (3.1) and obtains the regulated stress-energy tensor as in (3.6). Expanding $T_{ij}^{\rm reg}[\gamma]$ in ϵ (keeping $g_{(0)}$ fixed) we find that there is a logarithmic divergence,

$$T_{ij}^{\rm reg}[\gamma; \log] = \frac{1}{8\pi G_{\rm N}} \log \epsilon \; (\frac{3}{2} d - 1) h_{(d)ij}. \tag{C.1}$$

On the other hand, one can derive $T_{ij}^{\rm reg}[\gamma]$ starting from (3.2). One has to first rewrite the terms in (3.2) in terms of the induced metric. This is done in the previous appendix. Once $T_{ij}^{\rm reg}[\gamma]$ has been derived, we expand in ϵ. We find the following logarithmic divergence

$$T_{ij}^{\rm reg}[\gamma; \log] = \frac{1}{8\pi G_{\rm N}} \log \epsilon \left((1-d) h_{(d)ij} - T_{ij}^a, \right) \tag{C.2}$$

where T_{ij}^a is the stress-energy tensor of the action $\int {\rm d}^d x \sqrt{\det g_{(0)}}\, a_{(d)}$. If follows that

$$h_{(d)ij} = -\frac{2}{d} T_{ij}^a \tag{C.3}$$

We have also explicitly verified this relation by brute-force computation in $d = 4$.

D Asymptotic solution of the scalar field equation

We give here the first two orders of the solution of the equation (5.4)

$$\begin{aligned}
\phi_{(2)} &= \frac{1}{2(2\Delta - d - 2)} \left(\Box_0 \phi_{(0)} + (d - \Delta) \phi_{(0)} {\rm Tr}\, g_{(2)} \right), \\
\phi_{(4)} &= \frac{1}{4(2\Delta - d - 4)} \Bigg(\Box_0 \phi_{(2)} - 2\, {\rm Tr}\, g_{(2)} \phi_{(2)} - \frac{1}{2} (d - \Delta) \left[{\rm Tr}\, g_{(2)}^2 \, \phi_{(0)} - 2 {\rm Tr}\, g_{(2)}\, \phi_{(2)} \right] \\
&\qquad - \frac{1}{\sqrt{g_{(0)}}} \partial_\mu (\sqrt{g_{(0)}}\, g_{(2)}^{\mu\nu} \partial_\nu \phi_{(0)}) + \frac{1}{2} \partial^i {\rm Tr}\, g_{(2)} \partial_j \phi_{(0)} \Bigg),
\end{aligned} \tag{D.1}$$

where in $\Box_0$ the covariant derivatives are with respect to $g_{(0)}$.

If $2\Delta - d - 2k = 0$ one needs to introduce a logarithmic term in order for the equations to have a solution, as discussed in the main text. For instance, when $\Delta = \frac{1}{2} d + 1$, $\phi_{(2)}$ is undetermined, but instead one obtains for the coefficient of the logarithmic term,

$$\psi_{(2)} = -\frac{1}{4} \left(\Box_0 \phi_{(0)} + (\frac{d}{2} - 1)\, \phi_{(0)} {\rm Tr}\, g_{(2)} \right). \tag{D.2}$$

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